Elements of Linear Algebra

A matrix is any rectangular array of nubmer

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} column \\ vector \end{matrix}, \begin{bmatrix} 3, & 4 \end{bmatrix} \begin{matrix} row \\ vector \end{matrix}$$

if A has m rows and n columns then

$$A = \text{row } i \overset{\text{col } j}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & a_{ij} & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}$$

the order of the matrix A is mxn.

$$A = B \Leftrightarrow a_{ij} = b_{ij}$$

$\begin{bmatrix} 0, & 0, & \ldots, & 0 \end{bmatrix}$ zero row vector.

Scalar product of two vectors:

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad u^T v = u_1 v_1 + \ldots + u_n v_n$$

Sum of two or more vectors:

$$a + b = c, c_i = a_i + b_i, \ c = \begin{bmatrix} c_1 \\ \cdots \\ c_i \\ \cdots \\ c_n \end{bmatrix}$$

Scalar Multiples of a vector:

$$\alpha a = \begin{bmatrix} \alpha a_1 \\ \cdots \\ \alpha a_i \\ \cdots \\ \alpha a_n \end{bmatrix}$$

Linear Combination of Vectors:

$$\alpha_1 a_1 + \ldots + \alpha_i a_i + \ldots + \alpha_n a_n = 0$$

Addition of two Matrices:

$$A = \begin{bmatrix} a_{11} & & a_{1n} \\ & a_{ij} & \\ a_{m1} & & a_{mn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & & b_{1n} \\ & b_{ij} & \\ b_{m1} & & b_{mn} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & & c_{in} \\ & c_{ij} & \\ c_{m1} & & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{ij} + b_{ij}$$

The scalar multiple of a matrix:

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \cdots & \alpha a_{11} \\ \vdots & & \vdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{bmatrix}$$

The Transpose of a Matrix:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\Downarrow$$

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

Matrix Multiplication:

$$A = {}_r \overset{n}{\begin{bmatrix} a_{11} \dots & a_{1r} \\ a_{i1} \dots & a_{in} \\ a_{r1} \dots & a_{rn} \end{bmatrix}}, \quad B = {}_n \overset{s}{\begin{bmatrix} b_{11} & \cdots b_{1j} \cdots & b_{1s} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots b_{nj} \cdots & b_{ns} \end{bmatrix}}$$

$$AB = {}_r \overset{s}{\begin{bmatrix} \\ c_{ij} \\ \\ \end{bmatrix}}$$

$$c_{ij} = (a_i, b_j) = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

$$ABC = (AB)C = A(BC)$$

$$(AB)^T = B^T A^T$$

$$A(B + C) = AB + AC$$

$$A = \alpha_1 A_1 + \dots + \alpha_n A_n$$

Matrices & Systems of Linear Equations

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

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$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$$Ax = b$$

$$\begin{matrix} & n & & x & & b \\ m & \begin{bmatrix} \quad A \quad \end{bmatrix} & \cdot & [\;] & = & [\;] \end{matrix}$$

A solution to a linear system is a set of vectors $x = (x_1, \dots, x_n)$ that satisfies each of the system's equations.

$$x = (x_1, \dots, x_n) :$$

$$x_1A_1 + \dots + x_jA_j + .. + x_nA_n = b$$

Gauss-Jordan elimination

$$\begin{array}{c} y_1 = \\ y_2 = \\ y_3 = \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \quad a_{22} - pivot$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 + \frac{1}{a_{22}}y_2 - \frac{a_{23}}{a_{22}}x_3$$

$$y_1 = a_{11}x_1 + a_{12}\left(-\frac{a_{21}}{a_{22}}x_1 + \frac{1}{a_{22}}y_2 - \frac{a_{23}}{a_{22}}x_3\right) + a_{13}x_3 =$$

$$= \left(a_{11} - \frac{a_{12}a_{21}}{a_{22}}\right)x_1 + \frac{a_{12}}{a_{22}}y_2 + \left(a_{13} - \frac{a_{12}a_{23}}{a_{22}}\right)x_3 =$$

$$= \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}}x_1 + \frac{a_{12}}{a_{22}}y_2 + \frac{a_{13}a_{22} - a_{12}a_{23}}{a_{22}}x_3$$

$$\Delta_{11} = a_{11}a_{22} - a_{12}a_{21},\ \Delta_{13} = a_{13}a_{22} - a_{12}a_{23}$$

$$\Delta_{31} = a_{31}a_{22} - a_{21}a_{32},\ \Delta_{33} = a_{33}a_{22} - a_{23}a_{32}$$

$$\begin{array}{c} \\ y_1 \\ x_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & y_2 & x_3 \end{array} \\ \left[\begin{array}{ccc} \Delta_{11} & a_{12} & \Delta_{13} \\ -a_{21} & 1 & -a_{23} \\ \Delta_{31} & a_{32} & \Delta_{33} \end{array}\right] \end{array} : a_{22}$$

Application of G-J elimination

1 Inverse Matrix

$$Let\ A : \Re^n \rightarrow \Re^n,\ \ \det A \neq 0$$

$$A^{-1}\ exists$$

B is an inverse to A if

$$BA = AB = I = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}.$$

$$A = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \end{array}$$

$$\Downarrow$$

$$\begin{array}{c} \\ x_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \end{array}$$

$$\Downarrow$$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\Downarrow$$

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \end{array} = A^{-1}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1};$$

$$B^{-1}A^{-1}AB = I$$
$$\Downarrow$$
$$B^{-1}A^{-1} = (AB)^{-1}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

$$\left(A^T\right)^{-1} A^T = I \Rightarrow \left(\left(A^T\right)^{-1} A^T\right)^T = I$$

$$\left(\left(A^T\right)^{-1} A^T\right)^T = \left(A^T\right)^T \left(\left(A^T\right)^{-1}\right)^T = I$$

$$A\left(A^{T-1}\right)^T = I$$

$$\left(A^{T-1}\right)^T = A^{-1}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Systems of equations

$$x_1 + \quad + x_3 = 3$$

$$x_1 + x_2 + x_3 = 6$$

$$2x_1 + x_2 + 3x_3 = 13$$

$$\begin{matrix} \\ 3 = \\ 6 = \\ 13 = \end{matrix} \begin{matrix} x_1 \; x_2 \; x_3 \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \end{matrix} \Rightarrow \begin{matrix} \\ x_1 = \\ x_2 = \\ x_3 = \end{matrix} \begin{matrix} 3 \;\; 6 \;\; 13 \\ \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \end{matrix}$$

$$x_1 = 2 \cdot 3 + 1 \cdot 6 + (-1)13 = -1$$

$$x_2 = (-1)3 + 1 \cdot 6 + 0 \cdot 13 = 3$$

$$x_3 = (-1)3 + (-1)6 + 1 \cdot 13 = 4$$

$$
\begin{array}{rrrrr}
 & x_1 & x_2 & x_3 & 1 \\
0 = & 1 & 0 & 1 & -3 \\
0 = & 1 & 1 & 1 & -6 \\
0 = & 2 & 1 & 3 & -13
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{rrrrr}
 & 0 & x_2 & x_3 & 1 \\
x_1 = & 1 & 0 & -1 & 3 \\
0 = & 1 & 1 & 0 & -3 \\
0 = & 2 & 1 & 1 & -7
\end{array}
$$

$$\Downarrow$$

$$x_1 = -x_3 + 3$$

$$
\begin{array}{rrrr}
 & x_2 & x_3 & 1 \\
0 & 1 & 0 & -3 \\
0 & 1 & 1 & -7
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{rrrr}
 & 0 & x_3 & 1 \\
x_2 & 1 & 0 & 3 \\
0 & 1 & 1 & -4
\end{array}
$$

$$\Downarrow \qquad \Downarrow$$

$$0 = x_3 - 4 \Rightarrow x_3 = 4 \qquad x_2 = 0x_3 + 3 = 3$$

$$x_1 = -4 + 3 = -1$$

Matrix Rank

	x_1	x_2	x_3	x_4
y_1	1	2	3	4
y_2	2	1	6	8
y_3	3	6	9	12

⇓

	y_1	x_2	x_3	x_4
x_1	1	–2	–3	–4
y_2	2	–3	0	0
y_3	3	0	0	0

⇓

	y_1	y_2	x_3	x_4
x_1	1	–2	9	12
x_2	–2	1	0	0
y_3	–9	0	0	0

: (–3)

rank A = 2

steel	0.3	0.45	0.40
cars	0.15	0.20	0.10
mach	0.40	0.10	0.45

$= A$

$$A\begin{bmatrix} s \\ c \\ m \end{bmatrix} = \begin{bmatrix} a_{11}s + a_{12}c + a_{13}m \\ a_{21}s + a_{22}c + a_{23}m \\ a_{31}s + a_{32}c + a_{33}m \end{bmatrix} \quad \textit{consumption}$$

$$\begin{bmatrix} s \\ c \\ m \end{bmatrix} - A\begin{bmatrix} s \\ c \\ m \end{bmatrix} = \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix} - \textit{demand}$$

$$(I - A)\begin{bmatrix} s \\ c \\ m \end{bmatrix} = \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix}$$

↓

production

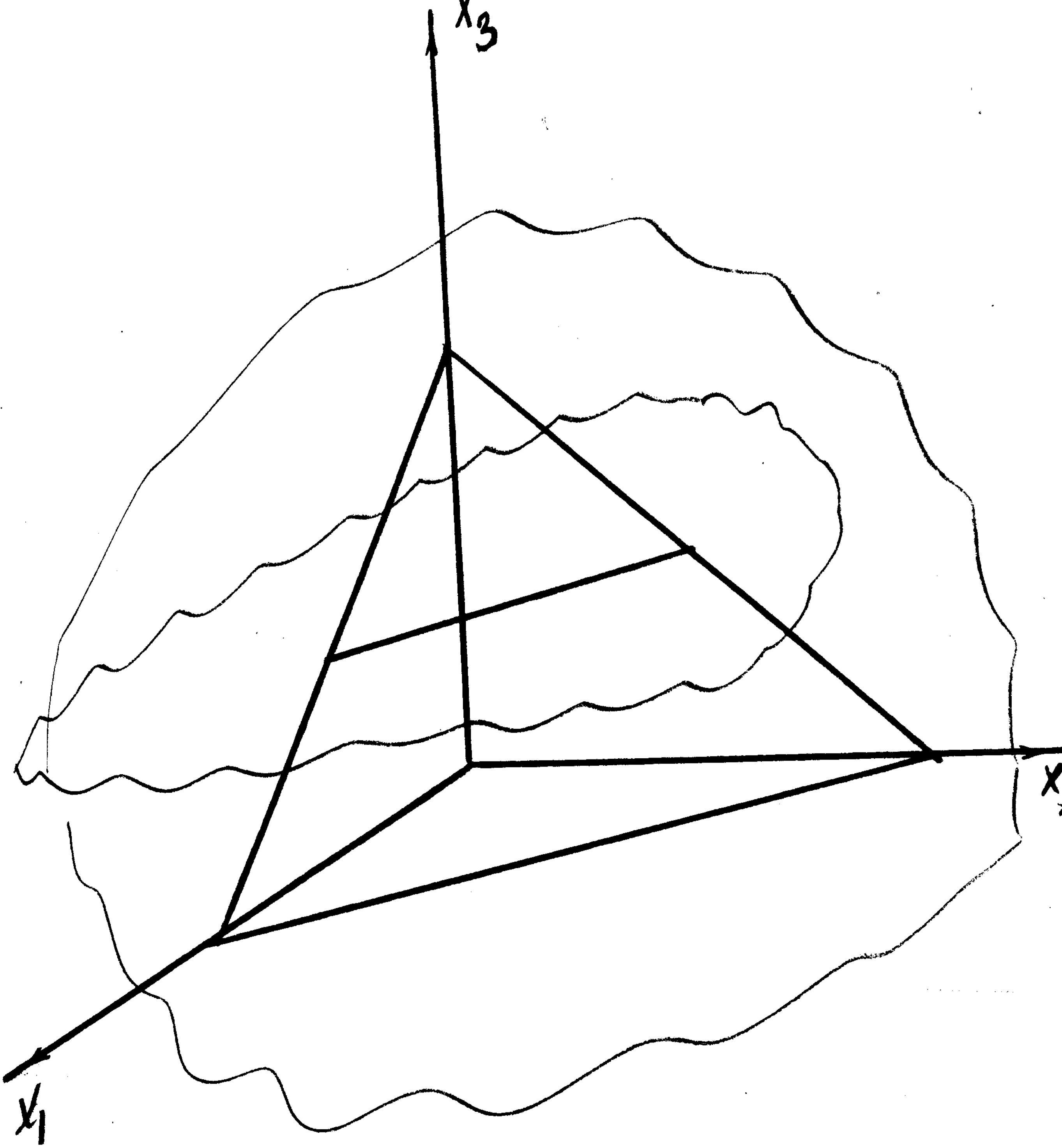
$$If \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix} \text{ is given, then}$$

$$y = \begin{bmatrix} s \\ c \\ m \end{bmatrix} = (I - A)^{-1} \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix}$$

$$\begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix} \Rightarrow \begin{bmatrix} d_s + 1 \\ d_c \\ d_m \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} \bar{s} \\ \bar{c} \\ \bar{m} \end{bmatrix} = (I - A)^{-1} \begin{bmatrix} d_s + 1 \\ d_c \\ d_m \end{bmatrix}$$

$$\bar{y} - y = (I - A)^{-1} \left(\begin{bmatrix} d_s + 1 \\ d_c \\ d_m \end{bmatrix} - \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix} \right) = (I - A)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

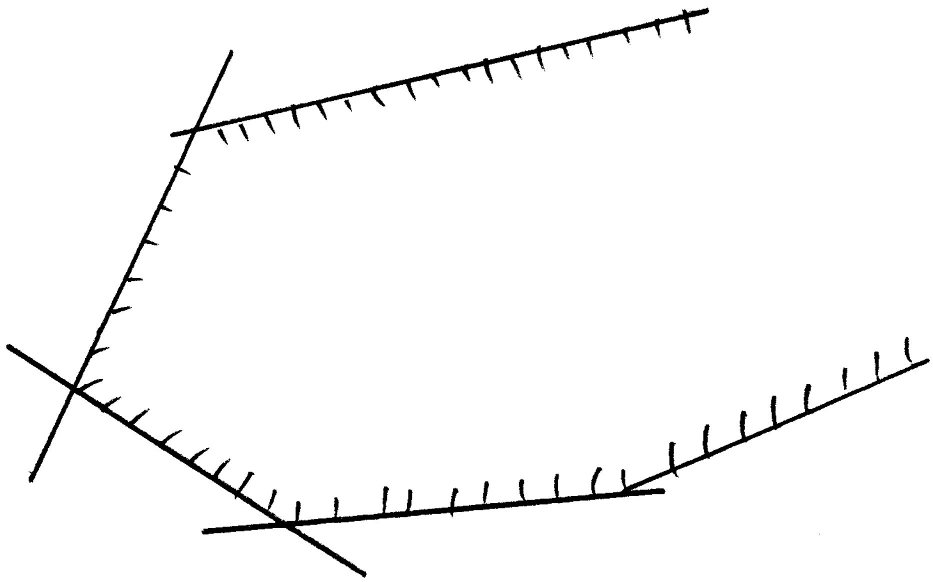


$$Z = C^T x \to \min$$

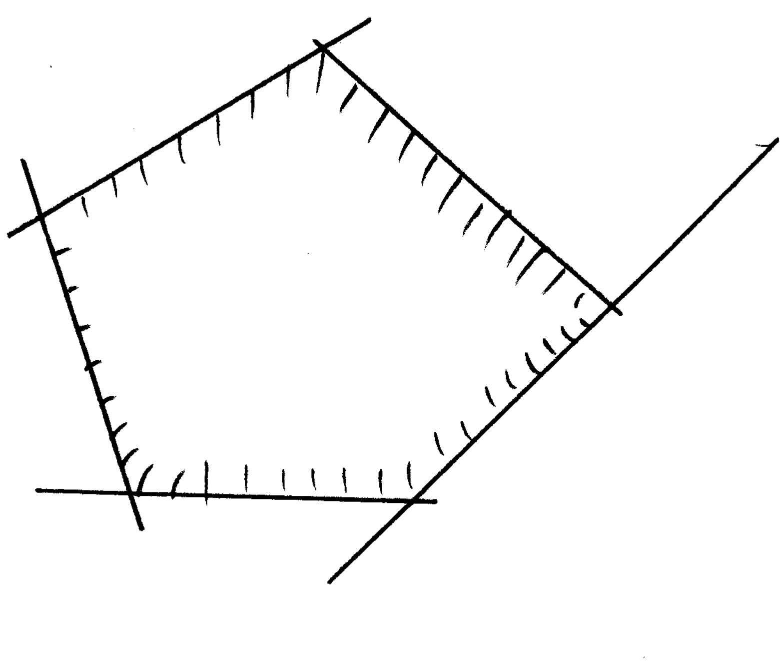
$$Ax = b$$

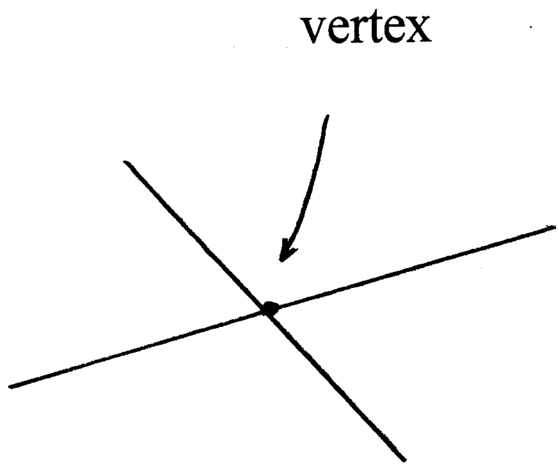
$$x \geq 0$$

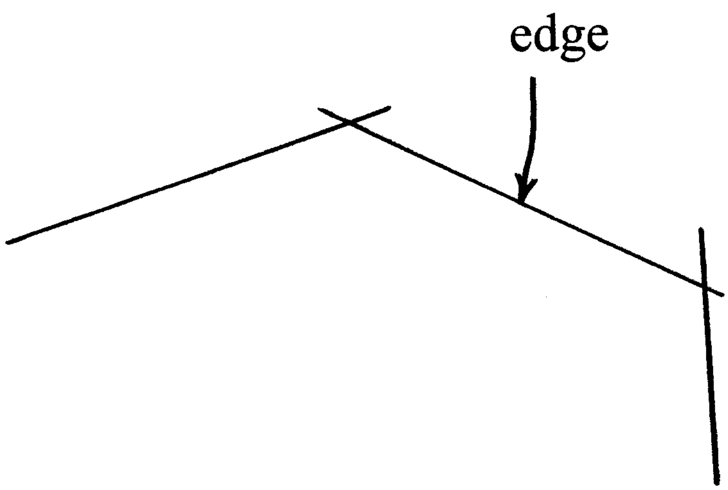
The intersection of half-spaces is called a polyhedron

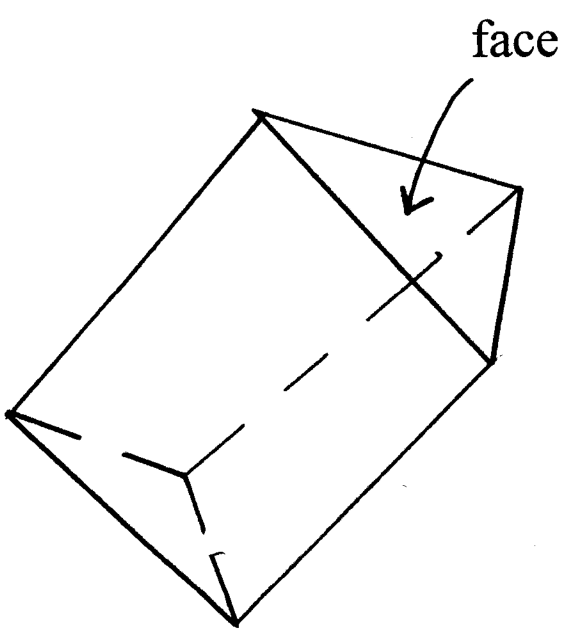


If a polyhedron is bounded then it is a polytop.

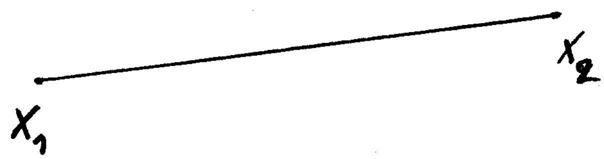



vertex


edge

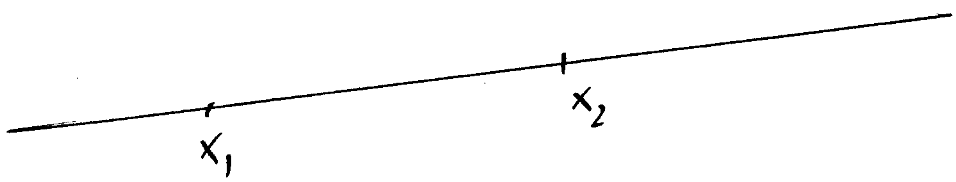

face

Convex combination of two points



$$x = \lambda_1 x_1 + \lambda_2 x_2\ ;\ \ \lambda_1 \geq 0,\ \ \lambda_2 \geq 0,\ \ \lambda_1 + \lambda_2 = 1$$

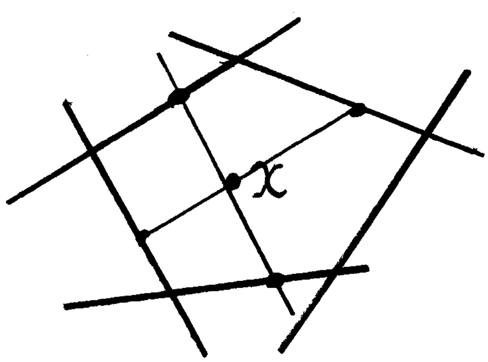
Affine combination



$$x = \lambda_1 x_1 + \lambda_2 x_2$$

Convex combination of m points

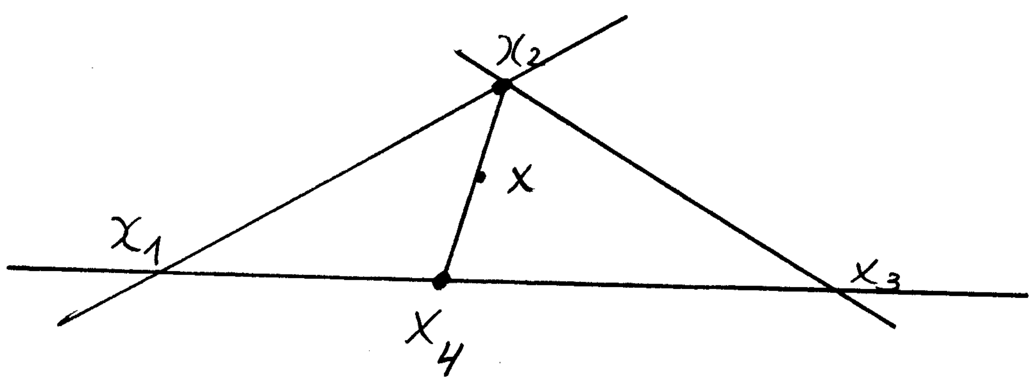
$$x = \sum_{i=1}^{n} \lambda_i x_i,\ \lambda_i \geq 0, \sum \lambda_i = 1$$

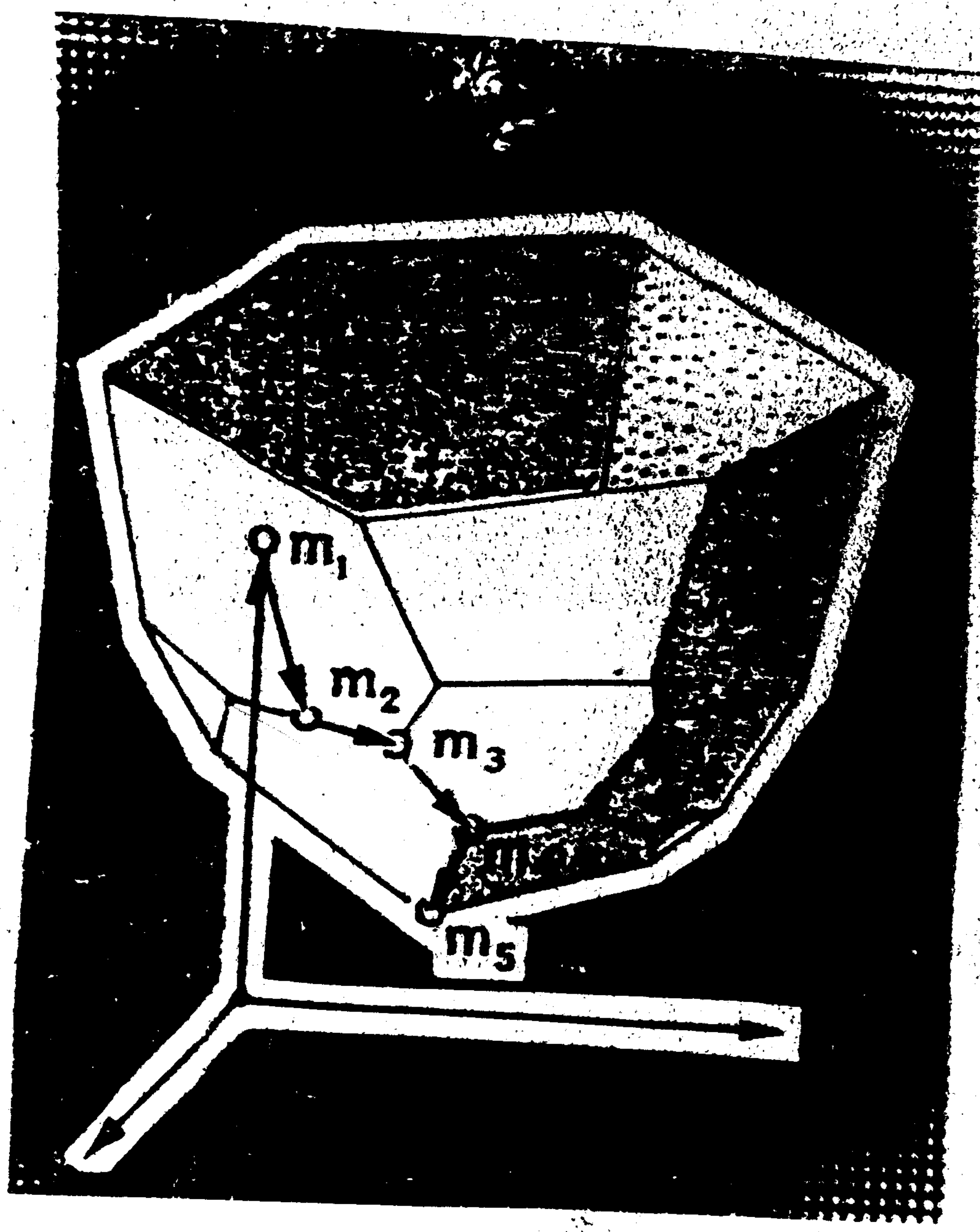


Vertex is a feasible point, which can't be represented as a convex combination of two different points in

Any point in a polytop can be represented as a convex combination of its vertices.

$$x \in \Omega; x = \sum_{i=1}^{n} \lambda_i x_i, \lambda_i \geq 0, \sum \lambda_i = 1$$




m1
m2
m3
m5